

# Multidimensional Markov FBSDEs with superquadratic growth

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December 12, 2016

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## ABSTRACT

We give local and global existence and uniqueness results for systems of coupled FBSDEs in the multidimensional setting and with generators allowed to grow arbitrarily fast in the control variable. Our results are based on Malliavin calculus arguments and pasting techniques.

**KEYWORDS:** Markovian FBSDE, superquadratic, existence and uniqueness

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## PAPER INFO

AMS CLASSIFICATION: (2000) 60H10, 60H07, 60H30

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## 1 Introduction

Given a multidimensional Brownian motion  $W$  on a probability space, consider the system of forward and backward stochastic differential equations

$$\begin{cases} X_t &= x + \int_0^t b_s(X_s, Y_s) ds + \int_0^t \sigma_s dW_s \\ Y_t &= h(X_T) + \int_t^T g_s(X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T] \end{cases} \quad (1.1)$$

where  $x$  is a known initial value,  $T > 0$  and  $b, \sigma, g$  and  $h$  are given functions. In this paper, we give conditions under which the system admits a unique solution in the case where the value process  $Y$  is multidimensional and the generator  $g$  can grow arbitrarily fast in the control process  $Z$ .

The focus in this paper is on Markovian systems, in which the functions  $b, \sigma, g$  and  $h$  are deterministic. We consider generators that are Lipschitz continuous in  $X$  and  $Y$  and locally Lipschitz continuous in  $Z$ . In this setting, and for one-dimensional value processes, the decoupled system (with  $b$  depending only on  $X$ ) has been solved by Cheridito and Nam [5] based on Malliavin calculus arguments. In fact, using that for Lipschitz continuous generators the trace of the Malliavin derivative of the value process  $Y$  is a version of the control process, they showed that the control process can be uniformly bounded, hence enabling solvability for locally Lipschitz generators by a truncation argument. We make ample use of their method to deal with the coupled system (1.1). In this case, we propose a Picard iteration scheme which yields a sequence that can

be proved to be a Cauchy sequence in an appropriate Banach space under uniform boundedness of the control processes derived using Malliavin calculus arguments provided that the time horizon is small enough. Moreover using the PDE representation of Markovian Lipschitz FBSDEs as developed for instance in Delarue [7] and a pasting procedure, we construct a unique global solution for generators with growth specified on the diagonal and under additional assumptions, mainly non-degeneracy of the volatility  $\sigma$ , see Theorem 2.2.

Systems such as (1.1) naturally appear in numerous areas of applied mathematics including stochastic control and mathematical finance, see Yong and Zhou [28], El Karoui et al. [11] and Horst et al. [16]. As shown for instance in Ma et al. [23] and Cheridito and Nam [5], in the Markovian case, FBSDEs can be linked to parabolic PDEs. More recently Fromm et al. [14] proved that FBSDEs can be used in the study of the Skorokhod embedding problem.

BSDEs and FBSDEs with Lipschitz continuous generators are well understood, we refer to El Karoui et al. [11] and Delarue [7]. If  $Y$  is one-dimensional and  $g$  is allowed to have quadratic growth in the control process  $Z$ , BSDEs' solutions have been obtained by Kobylanski [20], Barrieu and El Karoui [2] and Briand and Hu [3, 4] under various assumptions on the terminal condition  $\xi = h(X_T)$ . We further refer to Delbaen et al. [8], Drapeau et al. [10], Cheridito and Nam [5] and Heyne et al. [15] for results on one-dimensional BSDEs and FBSDEs with superquadratic growth. Mainly due to the absence of comparison principle, general solvability of multidimensional BSDEs with quadratic growth is less well understood. Some important progress have been achieved recently for BSDEs with small terminal conditions, see Tevzadze [26] and for more recent development, see Hu and Tang [17], Luo and Tangpi [22], Jamneshan et al. [19], Cheridito and Nam [6], Frei [12] and Xing and Žitković [27].

To the best of our knowledge, the only works studying well-posedness of coupled FBSDEs with quadratic growth are the article of Antonelli and Hamadène [1] and the preprints of Luo and Tangpi [22] and Fromm and Imkeller [13].

In [1] the authors consider a one-dimensional equation with one dimensional Brownian motion and impose monotonicity conditions on the coefficient so that comparison principles for SDEs and BSDEs can be applied. A (non-necessarily unique) solution is then obtained by monotone convergence of an iterative scheme. In [13], a fully coupled Markovian FBSDE is considered with multidimensional forward and value processes and locally Lipschitz generator in  $(Y, Z)$  and a existence of a unique local solution is obtained using the technique of decoupling fields and an extension to global solutions is proposed. Although the (non-Markovian) system studied in Luo and Tangpi [22] is the same as the one considered in the present paper, the techniques here are essentially different. Furthermore, the main results we present here can be extended to the non-Markovian setting and to random diffusion coefficient (when  $\sigma$  depends on  $X$  and  $Y$ ) under stronger assumptions involving the Malliavin derivatives of  $g$  and  $h$ . We refer to the Ph.D. thesis of Luo [21] for details.

In the next section, we present our probabilistic setting and the principal results of the paper. In Section 3 we prove local solvability of multidimensional BSDE with superquadratic growth and give conditions guaranteeing global solvability. Section 4 is dedicated to the proofs of the main results.

## 2 Main results

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$  be a filtered probability space, where  $(\mathcal{F}_t)_{t \in [0, T]}$  is the augmented filtration generated by a  $d$ -dimensional Brownian motion  $W$ , and  $\mathcal{F} = \mathcal{F}_T$  for a finite time horizon  $T \in (0, \infty)$ . The product  $\Omega \times [0, T]$  is endowed with the predictable  $\sigma$ -algebra. Subsets of  $\mathbb{R}^k$  and  $\mathbb{R}^{k \times k}$ ,  $k \in \mathbb{N}$ , are always endowed with the Borel  $\sigma$ -algebra induced by the Euclidean norm  $|\cdot|$ . The interval  $[0, T]$  is equipped with the Lebesgue measure. Unless otherwise stated, all equalities and inequalities between random variables and processes will be understood in the  $P$ -almost sure and  $P \otimes dt$ -almost sure sense, respectively. For  $p \in [1, \infty]$  and  $k \in \mathbb{N}$ , denote by  $\mathcal{S}^p(\mathbb{R}^k)$  the space of all predictable continuous processes  $X$  with values in  $\mathbb{R}^k$  such that  $\|X\|_{\mathcal{S}^p(\mathbb{R}^k)}^p := E[(\sup_{t \in [0, T]} |X_t|)^p] < \infty$ , and by  $\mathcal{H}^p(\mathbb{R}^k)$  the space of all predictable processes  $Z$  with values in  $\mathbb{R}^k$  such that  $\|Z\|_{\mathcal{H}^p(\mathbb{R}^k)}^p := E[(\int_0^T |Z_u|^2 du)^{p/2}] < \infty$ .

Let  $l, m \in \mathbb{N}$  be fixed. The solution of (1.1) with values in  $\mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^{l \times d}$  can be obtained under the following conditions:

(A1)  $b : [0, T] \times \mathbb{R}^l \rightarrow \mathbb{R}^m$  is a continuous function and there exist  $k_1, k_2, \lambda_1 \geq 0$  such that

$$|b_t(x, y) - b_t(x', y')| \leq k_1 |x - x'| + k_2 |y - y'| \text{ and } |b_t(x, y)| \leq \lambda_1(1 + |x| + |y|)$$

for all  $x, x' \in \mathbb{R}^m$  and  $y, y' \in \mathbb{R}^l$ .

(A2)  $\sigma : [0, T] \rightarrow \mathbb{R}^{m \times d}$  is a measurable function and there is  $\lambda_2 \geq 0$  such that  $|\sigma_t| \leq \lambda_2$  for all  $t \in [0, T]$ .

(A3)  $h : \mathbb{R}^m \rightarrow \mathbb{R}^l$  is a continuous function and there exists  $k_5 \geq 0$  such that

$$|h(x) - h(x')| \leq k_5 |x - x'|$$

for all  $x, x' \in \mathbb{R}^m$ .

(A4)  $g : [0, T] \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^{l \times d} \rightarrow \mathbb{R}^l$  is a continuous function satisfying  $g(0, 0, 0) \in L^2([0, T])$ , and there exist  $k_3, k_4 \geq 0$  and a nondecreasing function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|g_t(x, y, z) - g_t(x', y, z)| \leq k_3 |x - x'| \tag{2.1}$$

for all  $x, x' \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^l$  and  $z \in \mathbb{R}^{l \times d}$  such that  $|z| \leq M := 4\lambda_2 k_5 \sqrt{dl}$  and

$$|g_t(x, y, z) - g_t(x, y', z')| \leq k_4 |y - y'| + \rho(|z| \vee |z'|) |z - z'|$$

for all  $x \in \mathbb{R}^m$ ,  $y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$ .

(A5) There exists a constant  $K \geq 0$  such that

$$|g_t(x, y, z) - g_t(x', y, z) - g_t(x, y', z') + g_t(x', y', z')| \leq K |x - x'| (|y - y'| + |z - z'|)$$

for all  $t \in [0, T]$ ,  $x, x' \in \mathbb{R}^m$ ,  $y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$ .

Our first main result ensures local existence and uniqueness for the coupled FBSDE (1.1) under the previous assumptions. The proof is given in Section 4.

**Theorem 2.1.** *Assume that (A1)-(A5) hold. Then there exists a constant  $C_{k,\lambda,l,d} > 0$  depending only on  $k_1, k_2, k_3, k_4, k_5, \lambda_2, l$  and  $d$ , such that the FBSDE (1.1) has a unique solution  $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  with  $|Z_t| \leq M$ , whenever  $T \leq C_{k,\lambda,l,d}$ .*

Local existence results as Theorem 2.1 above have been obtained in [13, Theorem 3] and [22, Theorem 2.1] in essentially different settings and with different methods. A natural question is to find conditions under which the above result can be extended to obtain global solvability. Fromm and Imkeller [13] propose a concatenation procedure allowing to prove existence of solutions to a fully coupled FBSDE on a "maximal interval". In the present setting, under additional assumptions, a pasting method based on PDEs allows to get global existence and uniqueness for the FBSDE (1.1). The proof is also given in Section 4.

(A4')  $g : [0, T] \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^{l \times d} \rightarrow \mathbb{R}^l$  is a continuous function satisfying  $g(0, 0, 0) \in L^2([0, T])$ , and there exist  $k_3, k_4 \geq 0$  and a nondecreasing function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|g_t^i(x, y, z) - g_t^i(x', y', z')| \leq k_3 |x - x'| + k_4 |y - y'| + \rho(|z| \vee |z'|) |z^i - (z')^i|$$

for all  $i = 1, \dots, l; x, x' \in \mathbb{R}^m, y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$ .

**Theorem 2.2.** *Assume that (A1), (A2), (A3), (A4'), and (A5) hold and there exist  $\lambda_3 > 0; \lambda_4, \lambda_5 \geq 0$  such that<sup>1</sup>*

$$\begin{cases} |b_t(x, y)| & \leq \lambda_1(1 + |y|) \\ \langle x, \sigma_t \sigma_t^* x \rangle & \geq \lambda_3 |x|^2 \\ |g_t(x, y, z)| & \leq \lambda_4(1 + |y| + \rho(|z|) |z|) \\ |h(x)| & \leq \lambda_5 \end{cases} \quad (2.2)$$

for all  $t \in [0, T], x, x' \in \mathbb{R}^m, y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$ , then the FBSDE (1.1) has a unique global solution  $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  such that  $|Z_t| \leq \bar{M}$  for some constant  $\bar{M} \geq 0$ .

*Remark 2.3.* The following counterexample shows that the additional conditions in Theorem 2.2 cannot be dropped without violating global solvability. Consider the FBSDE

$$\begin{cases} X_t = \int_0^t Y_u du \\ Y_t = \int_t^T k X_u du - \int_t^T Z_u dW_u. \end{cases}$$

This equation can be rewritten as

$$Y_t = \int_t^T \int_0^s k Y_u du ds - \int_t^T Z_u dW_u. \quad (2.3)$$

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<sup>1</sup>  $\sigma_t^*$  is the transpose matrix of  $\sigma_t$ .

It has been shown in [9, Example 3.2] that if  $T\sqrt{k} < \frac{\pi}{2}$  then the BSDE with time-delayed generator (2.3) has a unique solution whereas if  $T\sqrt{k} = \frac{\pi}{2}$ , (2.3) may not have any solutions and if it does have one, there are infinitely many.  $\blacklozenge$

Theorem 2.2 extends to fully coupled generators provided that a more special structure is assumed. The proof of the next proposition is given in Section 4.3.

**Proposition 2.4.** *If there exist an invertible matrix  $\Gamma \in \mathbb{R}^{n \times n}$  and  $\lambda_3 > 0$ ;  $\lambda_4, \lambda_5 \geq 0$  such that (A1), (A2), (A3), (A4'), (A5) and (2.2) hold for  $\tilde{h} := \Gamma h$ ,  $\tilde{g}_t(x, y, z) := \Gamma g_t(x, \Gamma^{-1}y, \Gamma^{-1}z)$  and  $\tilde{b}_t(x, y) := b_t(x, \Gamma^{-1}y)$ , then the FBSDE (1.1) has a unique global solution  $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^2) \times \mathcal{S}^\infty(\mathbb{R}^{2 \times d})$  such that  $|Z_t| \leq \bar{M}$  for some constant  $\bar{M} \geq 0$ .*

The point of the above proposition is that global solvability can be obtained for a generator that does not satisfy the "diagonally superquadratic" growth condition (A4') provided that the transformed generator  $\tilde{g}$  does. Moreover, notice that Theorem 2.2 and Proposition 2.4 guarantee existence of a decoupling field, see [13]. In particular, the boundedness of  $Z$  guarantees a uniform Lipschitz property of the decoupling field.

Theorem 2.1 relies on an existence result for multidimensional BSDEs presented in Nam [24] and revisited in the next section.

### 3 Multidimensional BSDEs with bounded Malliavin derivatives

Let us introduce the spaces of Malliavin differentiable random variables and stochastic processes  $\mathcal{D}^{1,2}(\mathbb{R}^l)$  and  $\mathcal{L}_a^{1,2}(\mathbb{R}^l)$ . For a thorough treatment of the theory of Malliavin calculus we refer to Nualart [25]. Let  $\mathcal{M}$  be the class of smooth random variables  $\xi = (\xi^1, \dots, \xi^l)$  of the form

$$\xi^i = \varphi^i \left( \int_0^T h_s^{i1} dW_s, \dots, \int_0^T h_s^{in} dW_s \right)$$

where  $\varphi^i$  is in the space  $C_p^\infty(\mathbb{R}^n; \mathbb{R})$  of infinitely continuously differentiable functions whose partial derivatives have polynomial growth,  $h^{i1}, \dots, h^{in} \in L^2([0, T]; \mathbb{R}^d)$  and  $n \geq 1$ . For every  $\xi$  in  $\mathcal{M}$  let the operator  $D = (D^1, \dots, D^d) : \mathcal{M} \rightarrow L^2(\Omega \times [0, T]; \mathbb{R}^d)$  be given by

$$D_t \xi^i := \sum_{j=1}^n \frac{\partial \varphi^i}{\partial x_j} \left( \int_0^T h_s^{i1} dW_s, \dots, \int_0^T h_s^{in} dW_s \right) h_t^{ij}, \quad 0 \leq t \leq T, \quad 1 \leq i \leq l,$$

and the norm  $\|\xi\|_{1,2} := (E[|\xi|^2] + \int_0^T |D_t \xi|^2 dt)^{1/2}$ . As shown in Nualart [25], the operator  $D$  extends to the closure  $\mathcal{D}^{1,2}(\mathbb{R}^l)$  of the set  $\mathcal{M}$  with respect to the norm  $\|\cdot\|_{1,2}$ . A random variable  $\xi$  is Malliavin differentiable if  $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  and we denote by  $D_t \xi$  its Malliavin derivative. Denote by  $\mathcal{L}_a^{1,2}(\mathbb{R}^l)$  the space of processes  $Y \in \mathcal{H}^2(\mathbb{R}^l)$  such that  $Y_t \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  for all  $t \in [0, T]$ , the process  $DY_t$  admits a square integrable progressively measurable version and

$$\|Y\|_{\mathcal{L}_a^{1,2}(\mathbb{R}^l)}^2 := \|Y\|_{\mathcal{H}^2(\mathbb{R}^l)}^2 + E \left[ \int_0^T \int_0^T |D_r Y_t|^2 dr dt \right] < \infty.$$

We next consider a system of superquadratic BSDEs of the form

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u) du - \int_t^T Z_u dW_u \quad (3.1)$$

satisfying the following conditions:

- (B1)  $g : \Omega \times [0, T] \times \mathbb{R}^l \times \mathbb{R}^{l \times d} \rightarrow \mathbb{R}^l$  is a measurable function and there exist a constant  $B \in \mathbb{R}_+$  and a nondecreasing function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|g_t(y, z) - g_t(y', z')| \leq B |y - y'| + \rho(|z| \vee |z'|) |z - z'|$$

for all  $t \in [0, T]$ ,  $y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$ .

- (B2)  $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  and there exist constants  $A_{ij} \geq 0$  such that  $|D_t^j \xi^i| \leq A_{ij}$  for all  $i = 1, \dots, l$ ,  $j = 1, \dots, d$  and  $t \in [0, T]$ .

- (B3)  $g(0, 0) \in \mathcal{H}^4(\mathbb{R}^l)$  and there exist Borel-measurable functions  $q_{ij} : [0, T] \rightarrow \mathbb{R}_+$  satisfying  $\int_0^T q_{ij}^2(t) dt < \infty$  and for every pair  $(y, z) \in \mathbb{R}^l \times \mathbb{R}^{l \times d}$  with

$$|z| \leq Q := \sqrt{2 \sum_{j=1}^d \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_0^T q_{ij}^2(t) dt \right)}$$

it holds

- $g(y, z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^l)$  with  $|D_u^j g_t^i(y, z)| \leq q_{ij}(t)$  for all  $i = 1, \dots, l$ ,  $j = 1, \dots, d$  and  $u \in [0, T]$ ,
- for almost all  $u \in [0, T]$  one has

$$|D_u g_t(y, z) - D_u g_t(y', z')| \leq K_u(t) (|y - y'| + |z - z'|)$$

for all  $t \in [0, T]$ ,  $y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$  for some  $\mathbb{R}_+$ -valued adapted process  $(K_u(t))_{t \in [0, T]}$  satisfying  $\int_0^T \|K_u\|_{\mathcal{H}^4(\mathbb{R})}^4 du < \infty$ .

The following is an extension of Cheridito and Nam [5, Theorem 2.2] to the multidimensional case. It was proved in Nam [24] under slightly different assumptions. We give the proof for the sake of completeness.

**Theorem 3.1.** Assume that (B1)-(B3) hold and  $T \leq \frac{\log(2)}{2B + \rho^2(Q) + 1}$ . Then the BSDE (3.1) admits a unique solution in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  and  $|Z_t| \leq Q$ .

Consider the following stronger versions of the conditions (B1) and (B3):

- (B1')  $g$  is continuously differentiable in  $(y, z)$  and there exist constants  $B \in \mathbb{R}_+$  and  $\rho \in \mathbb{R}_+$  such that  $|\partial_y g_t(y, z)| \leq B$  and  $|\partial_z g_t(y, z)| \leq \rho$  for all  $t \in [0, T]$ ,  $y, y' \in \mathbb{R}^l$  and  $z, z' \in \mathbb{R}^{l \times d}$ .

(B3') The condition (B3) holds for all  $(y, z) \in \mathbb{R}^l \times \mathbb{R}^{l \times d}$ .

**Lemma 3.2.** *If (B1'), (B2) and (B3') hold, then the BSDE (3.1) admits a unique solution  $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l \times d})$  and*

$$|Z_t^j|^2 \leq \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_t^T q_{ij}^2(s) e^{-(2B+\rho^2+1)(T-s)} ds \right) e^{(2B+\rho^2+1)(T-t)} \quad \text{for all } j = 1, \dots, d. \quad (3.2)$$

*Proof.* By Cheridito and Nam [5, Lemma 2.5], the condition (B2) implies  $E[|\xi|^p] < +\infty$  for all  $p \in [1, \infty)$ . It follows from El Karoui et al. [11, Theorem 5.1 and Proposition 5.3] that the BSDE (3.1) has a unique solution  $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l \times d})$ , which is Malliavin differentiable. Moreover for every  $i = 1, \dots, l$  and  $j = 1, \dots, d$ , the process  $(D_r^j Y_t^i, D_r^j Z_t^i)_{t \in [0, T]}$  has a version  $(U_t^{ij,r}, V_t^{ij,r})_{t \in [0, T]}$  which satisfies

$$U_t^{ij,r} = 0, \quad V_t^{ij,r} = 0, \quad \text{for } 0 \leq t < r \leq T,$$

and is the unique solution in  $\mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$  of the BSDE

$$U_t^{j,r} = D_r^j \xi + \int_t^T \partial_y g_s(Y_s, Z_s) U_s^{j,r} + \partial_z g_s(Y_s, Z_s) V_s^{j,r} + D_r^j g_s(Y_s, Z_s) ds - \int_t^T V_s^{j,r} dW_s.$$

Applying Itô's formula to  $|U_t^{j,r}|^2$  yields

$$\begin{aligned} |U_t^{j,r}|^2 &= |D_r^j \xi|^2 - \int_t^T 2U_s^{j,r} V_s^{j,r} dW_s \\ &\quad + \int_t^T 2U_s^{j,r} \partial_y g_s(Y_s, Z_s) U_s^{j,r} + 2U_s^{j,r} \partial_z g_s(Y_s, Z_s) V_s^{j,r} + 2U_s^{j,r} D_r^j g_s(Y_s, Z_s) - |V_s^{j,r}|^2 ds \\ &\leq |D_r^j \xi|^2 - \int_t^T 2U_s^{j,r} V_s^{j,r} dW_s + \int_t^T 2B|U_s^{j,r}|^2 + 2\rho|U_s^{j,r}||V_s^{j,r}| + 2\sqrt{\sum_{i=1}^l q_{ij}^2(s)} |U_s^{j,r}| - |V_s^{j,r}|^2 ds \\ &\leq |D_r^j \xi|^2 - \int_t^T 2U_s^{j,r} V_s^{j,r} dW_s + \int_t^T (2B + \rho^2 + 1) |U_s^{j,r}|^2 + \sum_{i=1}^l q_{ij}^2(s) ds. \end{aligned}$$

Using condition (B3) and taking conditional expectation in the above inequality yields

$$|U_t^{j,r}|^2 \leq E \left[ \sum_{i=1}^l A_{ij}^2 + \int_t^T (2B + \rho^2 + 1) |U_s^{j,r}|^2 + \sum_{i=1}^l q_{ij}^2(s) ds \middle| \mathcal{F}_t \right]. \quad (3.3)$$

By El Karoui et al. [11, Proposition 5.3] the process  $Z$  is a version of the trace  $(U_t^t)_{t \in [0, T]}$  of the Malliavin derivative of  $Y$ . Hence (3.2) follows from (3.3) by applying Gronwall's inequality.  $\square$

*Proof (Theorem 3.1).* Define the Lipschitz continuous function  $\tilde{g}$  by

$$\tilde{g}_t(y, z) = \begin{cases} g_t(y, z) & \text{if } |z| \leq Q, \\ g_t(y, Qz/|z|) & \text{if } |z| > Q. \end{cases} \quad (3.4)$$

By Cheridito and Nam [5, Lemma 2.5] and El Karoui et al. [11, Theorem 5.1] the BSDE corresponding to  $(\tilde{g}, \xi)$  has a unique solution  $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l \times d})$ . For  $x = (y, z) \in \mathbb{R}^{l+l \times d}$  let  $\beta \in C^\infty(\mathbb{R}^{l+l \times d})$  be the mollifier

$$\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant  $\lambda \in \mathbb{R}_+$  is chosen such that  $\int_{\mathbb{R}^{l+l \times d}} \beta(x) dx = 1$ . Set  $\beta^n(x) := n^{l+l \times d} \beta(nx)$ ,  $n \in \mathbb{N} \setminus \{0\}$ , and define

$$g_t^n(\omega, x) := \int_{\mathbb{R}^{l+l \times d}} \tilde{g}_t(\omega, x') \beta^n(x - x') dx'$$

so that for each  $n > 0$  the function  $g^n$  satisfies (B1') and (B3') with the constant  $\rho$  replaced by  $\rho(Q)$ . By Lemma 3.2 the BSDE corresponding to  $(g^n, \xi)$  has a unique solution  $(Y^n, Z^n)$  in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l \times d})$  which satisfies

$$\begin{aligned} |Z_t^{n,j}|^2 &\leq \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_t^T q_{ij}^2(s) e^{-(2B+\rho^2(Q)+1)(T-s)} ds \right) e^{(2B+\rho^2(Q)+1)(T-t)} \\ &\leq \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_0^T q_{ij}^2(s) ds \right) e^{(2B+\rho^2(Q)+1)T}. \end{aligned}$$

Since  $T \leq \frac{\log(2)}{2B+\rho^2(Q)+1}$  we obtain

$$|Z_t^{n,j}|^2 \leq 2 \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_0^T q_{ij}^2(s) ds \right) \quad \text{for all } j = 1, \dots, d.$$

This shows  $|Z_t^n| \leq Q$ . Since  $g^n$  converges uniformly in  $(t, \omega, y, z)$  to  $\tilde{g}$ , using the procedure of the proof of Cheridito and Nam [5, Theorem 2.2], it follows that  $(Y^n, Z^n)$  converges to  $(Y, Z)$  in  $\mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$ , so that  $|Z_t| \leq Q$ . Since  $\tilde{g}(y, z) = g(y, z)$  for all  $(y, z) \in \mathbb{R}^l \times \mathbb{R}^{l \times d}$  with  $|z| \leq Q$ , it follows that  $(Y, Z)$  is the unique solution of the BSDE corresponding to  $(\xi, g)$  in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$ .  $\square$

**Corollary 3.3.** Suppose (B1)-(B3) hold,  $T \leq \frac{\log(2)}{2B+\rho^2(Q)+1}$  and  $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  is the solution of the BSDE (3.1). Then  $Y_t \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  for all  $t \in [0, T]$  and for every  $j = 1, \dots, d$ , one has

$$|D_r^j Y_t|^2 \leq 2 \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_0^T q_{ij}^2(s) ds \right) \quad \text{for all } r \in [0, t]. \quad (3.5)$$



*Proof.* Since  $|Z| \leq Q$  is bounded,  $(Y, Z)$  solves the BSDE with terminal condition  $\xi$  and generator  $\tilde{g}$  defined by (3.4). If  $g$  satisfies (B1') and (B3'), then the result follows from Lemma 3.2. Otherwise consider the sequence of smooth functions  $g^n$  converging to  $g$  as defined in the proof of Theorem 3.1. Let  $(Y^n, Z^n) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l \times d})$  be the solutions to the BSDEs corresponding to  $(g^n, \xi)$ , which converge to  $(Y, Z)$  in  $\mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$ . By Lemma 3.2  $(Y_t^n, Z_t^n) \in \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l \times d})$  for each  $t \in [0, T]$  and the arguments in the proof of Theorem 3.1 imply

$$|D_r^j Y_t^n|^2 \leq 2 \left( \sum_{i=1}^l A_{ij}^2 + \sum_{i=1}^l \int_0^T q_{ij}^2(s) ds \right) \quad j = 1, \dots, d, \quad r, t \in [0, T].$$

Hence,  $\sup_{n \in \mathbb{N}} E[\int_0^T |D_r^j Y_t^n|^2 dr] < \infty$  for each  $t \in [0, T]$ . Since  $(Y_t^n)$  converges to  $Y_t$  in  $L^2$ , it follows from Nualart [25, Lemma 1.2.3] that  $Y_t \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  and  $(DY_t^n)$  converges to  $DY_t$  in the weak topology of  $\mathcal{H}^2(\mathbb{R}^{l \times d})$ . Thus,  $D_r Y_t$  satisfies (3.5).  $\square$

As a consequence to Theorem 3.1, we give a condition for global solvability of fully coupled systems of BSDEs. For the remainder of this section we put

$$\Delta_n := \frac{\log(2)}{2B + \rho^2(2^n Q) + 1}, \quad n \in \mathbb{N}.$$

**Proposition 3.4.** *Assume that (B1)-(B2) hold, that there exists  $N \in \mathbb{N}$  such that  $\sum_{n=0}^N \Delta_n \geq T$ , and (B3) holds with  $Q$  replaced by  $2^N Q$ . Then the BSDE (3.1) has a unique solution in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  and  $|Z_t| \leq 2^N Q$ .*

*Proof.* If  $T \leq \Delta_0$  then the result follows from Theorem 3.1. Otherwise, if  $T > \Delta_0$  it follows by the same arguments as in the proof of Theorem 3.1 that the BSDE (3.1) has a unique solution  $(Y^0, Z^0)$  in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  on the interval  $[T - \Delta_0, T]$ . Moreover,  $Z^0$  satisfies  $|Z_t^0| \leq Q$  and by Corollary 3.3 one has  $Y_{T-\Delta_0}^0 \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  and for every  $r \leq T - \Delta_0$ ,

$$|D_r^j Y_{T-\Delta_0}^0|^2 \leq \sum_{i=1}^l 2|A_{ij}|^2 + \sum_{i=1}^l \int_0^T 2|q_{ij}(t)|^2 dt \quad \text{for all } j = 1, \dots, d.$$

Since  $g$  satisfies (B3) for all  $(y, z) \in \mathbb{R}^l \times \mathbb{R}^{l \times d}$  such that  $|z| \leq cQ$ , again by Theorem 3.1 the BSDE (3.1) with terminal condition  $Y_{T-\Delta_0}^0$  has a unique solution  $(Y^1, Z^1)$  in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  on  $[(T - \Delta_0 - \Delta_1) \vee 0, T - \Delta_0]$ , and

$$|D_r^j Y_{(T-\Delta_0-\Delta_1) \vee 0}^1|^2 \leq \sum_{i=1}^l 2^2 |A_{ij}|^2 + \sum_{i=1}^l \int_0^T (2^2 + 2)|q_{ij}(t)|^2 dt, \quad \text{for all } j = 1, \dots, d$$

$$|Z_t^1| \leq 2Q, \quad t \in [(T - \Delta_0 - \Delta_1) \vee 0, T - \Delta_0].$$

Repeating the previous arguments, for  $N \geq 2$  the BSDE (3.1) has a unique solution  $(Y^N, Z^N)$  in  $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  on  $[(T - \sum_{n=0}^N \Delta_n) \vee 0, (T - \sum_{n=0}^{N-1} \Delta_n) \vee 0]$  with terminal condition

$Y_{(T-\sum_{n=0}^{N-1} \Delta_n) \vee 0}$ . Moreover,

$$|D_r^j Y_{(T-\sum_{n=0}^N \Delta_n) \vee 0}^N|^2 \leq \sum_{i=1}^l 2^N |A_{ij}|^2 + \sum_{i=1}^l \int_0^T \left( \sum_{k=1}^N 2^k \right) |q_{ij}(t)|^2 dt \quad \text{for all } j = 1, \dots, d$$

$$|Z_t^N| \leq 2^N Q, \quad t \in \left[ (T - \sum_{n=0}^N \Delta_n) \vee 0, (T - \sum_{n=0}^{N-1} \Delta_n) \vee 0 \right].$$

Hence, the pair  $(Y, Z)$  given by

$$Y := Y^0 1_{[T-\Delta_1, T]} + \sum_{n=1}^N Y^n 1_{[(T-\sum_{i=0}^n \Delta_i) \vee 0, (T-\sum_{i=0}^{n-1} \Delta_i) \vee 0]}$$

$$Z := Z^0 1_{[T-\Delta_1, T]} + \sum_{n=1}^N Z^n 1_{[(T-\sum_{i=0}^n \Delta_i) \vee 0, (T-\sum_{i=0}^{n-1} \Delta_i) \vee 0]}$$

solves (3.1) and its uniqueness follows from Theorem 3.1.  $\square$

*Remark 3.5.* The condition  $\sum_{n=0}^N \Delta_n \geq T$  for some  $N \in \mathbb{N}$  does not guarantee global solvability of multidimensional BSDEs with superquadratic growth. In fact, if  $\rho(x) \geq C(1 + \sqrt{x})$  for all  $x \geq 0$ , then  $\sum_{n \geq 0} \Delta_n < \infty$ . However, it does guarantee global solvability for BSDEs whose generator grows slightly faster than the linear function. For instance, if  $\rho(x) \leq C(1 + \sqrt{\log(1+x)})$  one has

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\log(2)}{2B + \rho^2(2^N Q) + 1} &\geq \sum_{n=0}^{\infty} \frac{\log(2)}{2B + 2C^2(1 + \log(2^N(1+Q))) + 1} \\ &= \sum_{n=0}^{\infty} \frac{\log(2)}{2B + 2C^2(1 + \log(1+Q) + n \log(2)) + 1} = \infty. \quad \blacklozenge \end{aligned}$$

## 4 Coupled FBSDE with superquadratic growth

### 4.1 Proof of Theorem 2.1

*Step 1:* We first assume that  $h, b$  and  $g$  are continuously differentiable in all variables. Let us define

$$C_{k,\lambda,l,d}^1 := \frac{k_5^2}{k_3^2} \wedge \frac{\log 2}{k_1} \wedge \frac{\lambda_2}{k_2 M} \wedge \frac{\log 2}{2k_4 + \rho^2(M) + 1}$$

with  $M := 4k_5 \lambda_2 \sqrt{dl}$ . We will show that for  $T \leq C_{k,\lambda,l,d}^1$ , the sequence  $(X^n, Y^n, Z^n)$  given by  $X^0 = 0, Y^0 = 0, Z^0 = 0$  and

$$\begin{cases} X_t^{n+1} &= x + \int_0^t b(X_u^{n+1}, Y_u^n) du + \int_0^t \sigma_u dW_u \\ Y_t^{n+1} &= h(X_T^{n+1}) + \int_t^T g_u(X_u^{n+1}, Y_u^{n+1}, Z_u^{n+1}) du - \int_t^T Z_u^{n+1} dW_u, \quad n \geq 1 \end{cases}$$

is well defined and that  $|Z_t^n| \leq M$  for all  $n \in \mathbb{N}$  and  $t \in [0, T]$ . The process  $X^1$  is well defined,  $X_t^1$  belongs to  $\mathcal{D}^{1,2}(\mathbb{R}^m)$  for every  $t$  and the process  $(D_r X_t)_t \in [0, T]$  satisfies the linear equation

$$D_r X_t^1 = 0, \quad 0 \leq t < r \leq T,$$

$$D_r X_t^1 = \int_r^t (\partial_x b D_r X_u^1 + \partial_y b D_r Y_u^0) du + D_r \left( \int_r^t \sigma_u dW_u \right), \quad 0 \leq r \leq t \leq T,$$

with  $D_r(\int_r^t \sigma_u dW_u) = \sigma 1_{[r,t]}$ , see Nualart [25, Lemma 2.2.1 and Theorem 2.2.1]. Hence, since  $b$  is Lipschitz continuous, we have

$$|D_r X_t^1| \leq \int_r^t k_1 D_r X_u^1 du + \sigma_r \quad \text{and} \quad |D_r X_t^1| \leq \lambda_2 e^{T k_1},$$

where the second estimate comes from Gronwall's inequality. We will now show that since  $T \leq C_{k,\lambda,l,d}^1$ ,  $h(X_T^1)$  and  $g(X^1, \cdot, \cdot)$  satisfy (B1)-(B3). In fact, since  $h$  is continuously differentiable and  $X_T^1 \in \mathcal{D}^{1,2}(\mathbb{R}^m)$ , it follows from the chain rule, see for instance Nualart [25, Proposition 1.2.4], that  $h(X_T^1) \in \mathcal{D}^{1,2}(\mathbb{R}^l)$  and  $|D_r^j(h(X_T^1))| = |\partial_x h(X_T^1) D_r^j X_T^1| \leq \lambda_2 k_5 e^{T k_1}$  for all  $r \in [0, T]$ ,  $j = 1, \dots, d$ . Using  $T \leq \frac{\log 2}{k_1}$ , we deduce that  $h(X_T^1)$  satisfies (B2) with  $A_{ij} := 2\lambda_2 k_5$ . Similarly, by (A4) and using that the function  $x \mapsto g(x, y, z)$  is continuously differentiable, it follows that  $g(X_T^1, y, z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^l)$  and  $|D_r^j(g^i(X_T^1, y, z))| \leq \lambda_2 k_3 e^{T k_1}$ ,  $j = 1, \dots, d$  for all  $(y, z) \in \mathbb{R}^l \times \mathbb{R}^{l \times d}$  such that  $|z| \leq M$  and, due to (A5), applying the same argument to  $\hat{g}_t(x, y, y', z, z') := g_t(x, y, z) - g_t(x, y', z')$  yields

$$|D_r^j g_t(X_t^1, y, z) - D_r^j g_t(X_t^1, y', z')| \leq K \lambda_2 e^{T k_1}.$$

Using  $T \leq \frac{k_5^2}{k_3^2} \wedge \frac{\log 2}{k_1}$ , we deduce that  $g(X^1, y, z)$  satisfies (B3) with  $q_{ij} = 2\lambda_2 k_3$  and  $K_u(t) := 2K \lambda_2$ . Moreover due to (A4), the function  $(t, y, z) \mapsto g_t(X_t^1, y, z)$  satisfies (B1). Therefore, by  $T \leq \frac{\log 2}{2k_4 + \rho^2(M)+1}$ , Theorem 3.1 ensures that  $(Y^1, Z^1)$  exists. Consider the function  $\tilde{g}$  defined by

$$\tilde{g}_t(x, y, z) = \begin{cases} g_t(x, y, z) & \text{if } |z| \leq M \\ g_t(x, y, zM/|z|) & \text{if } |z| > M. \end{cases}$$

Since  $(Y^1, Z^1)$  also solves the BSDE with terminal condition  $h(X_T^1)$  and a Lipschitz generator  $\tilde{g}(X^1, \cdot, \cdot)$ , it follows from Lemma 3.2 and its proof that  $(Y_t^1, Z_t^1) \in \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l \times d})$  for all  $t \in [0, T]$  and  $D_t Y^1$  is bounded and it holds  $Z_t^1 = D_t Y_t^1$ . In addition, we have  $|D_r X_t^1| \leq 4\lambda_2$  and  $|D_r Y_t^1| \leq M$ .

Now let  $n \in \mathbb{N}$ , assume that  $(X_t^n, Y_t^n, Z_t^n) \in \mathcal{D}^{1,2}(\mathbb{R}^m) \times \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l \times d})$ ,  $Z_t^n = D_t Y_t^n$  and  $|D_r X_t^n| \leq 4\lambda_2$ ,  $|D_r Y_t^n| \leq M$  for all  $r, t \in [0, T]$ . The process  $X^{n+1}$  is well defined, for each  $t$ ,  $X_t^{n+1}$  belongs to  $\mathcal{D}^{1,2}(\mathbb{R}^m)$  and it holds

$$D_r X_t^{n+1} = 0, \quad 0 \leq t < r \leq T,$$

$$D_r X_t^{n+1} = \sigma_r + \int_r^t (\partial_x b D_r X_u^{n+1} + \partial_y b D_r Y_u^n) du, \quad 0 \leq r \leq t \leq T.$$

Since  $\partial_x b$ ,  $\partial_y b$  and  $\sigma$  are bounded by  $k_1$ ,  $k_2$  and  $\lambda_2$  respectively, it follows from Gronwall's inequality that

$$|D_r X_t^{n+1}| \leq e^{Tk_1} \left( \lambda_2 + k_2 \int_0^T |D_r Y_u^n| du \right).$$

Hence,

$$|D_r X_t^{n+1}| \leq e^{Tk_1} (\lambda_2 + k_2 TM) < \infty \quad (4.1)$$

so that since  $T \leq \frac{\lambda_2}{k_2 M}$ , we have  $|D_r X_t^{n+1}| \leq 4\lambda_2$ . As above,  $h(X_T^{n+1})$  and  $g(X^{n+1}, y, z)$  are Malliavin differentiable and satisfy (B1)-(B3) with  $A_{ij} := 2\lambda_2 k_5$ ,  $q_{ij} = 2\lambda_2 k_3$  and  $K_u(t) := 2K\lambda_2$ . It then follows again from Theorem 3.1 that  $(Y^{n+1}, Z^{n+1})$  exists and  $|Z^{n+1}| \leq M$  is bounded. Since  $(Y^{n+1}, Z^{n+1})$  also solves the BSDE with terminal condition  $h(X_T^{n+1})$  and a Lipschitz generator  $\tilde{g}(X^{n+1}, \cdot, \cdot)$ , Lemma 3.2 and its proof guarantee that  $(Y_t^{n+1}, Z_t^{n+1}) \in \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l \times d})$  for all  $t \in [0, T]$  and  $D_t Y^{n+1}$  is bounded and it holds  $Z_t^{n+1} = D_t Y_t^{n+1}$ , with  $|D_r Y_t^{n+1}| \leq M$ .

*Step 2:* Now we show that there is a positive constant  $2_{k,\lambda,l,d}^2$  such that if  $T \leq 2_{k,\lambda,l,d}^2$ , then  $(X^n, Y^n, Z^n)$  is a Cauchy sequence in  $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$ . Using (A1) we can estimate the norm of the difference  $X_t^{n+1} - X_t^n$  as

$$|X_t^{n+1} - X_t^n|^2 \leq 2 \left( \int_0^t k_1 |X_s^{n+1} - X_s^n| ds \right)^2 + 2 \left( \int_0^t k_2 |Y_s^n - Y_s^{n-1}| ds \right)^2.$$

Thus

$$\sup_{0 \leq t \leq T} |X_t^{n+1} - X_t^n|^2 \leq 2 \left( \int_0^T k_1 |X_s^{n+1} - X_s^n| ds \right)^2 + 2 \left( \int_0^T k_2 |Y_s^n - Y_s^{n-1}| ds \right)^2.$$

Taking expectation on both sides and using Cauchy-Schwarz' inequality, we have

$$\begin{aligned} E \left[ \sup_{0 \leq t \leq T} |X_t^{n+1} - X_t^n|^2 \right] &\leq 2Tk_1^2 E \left[ \int_0^T |X_s^{n+1} - X_s^n|^2 ds \right] + 2Tk_2^2 E \left[ \int_0^T |Y_s^n - Y_s^{n-1}|^2 ds \right] \\ &\leq 2T^2 k_1^2 E \left[ \sup_{0 \leq t \leq T} |X_t^{n+1} - X_t^n|^2 \right] + 2T^2 k_2^2 E \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^{n-1}|^2 \right]. \end{aligned}$$

Choosing  $T$  to be small enough so that  $2T^2 k_1^2 \leq \frac{1}{2}$ , it follows

$$E \left[ \sup_{0 \leq t \leq T} |X_t^{n+1} - X_t^n|^2 \right] \leq 4T^2 k_2^2 E \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^{n-1}|^2 \right]. \quad (4.2)$$

On the other hand, applying Itô's formula to  $e^{\beta t}|Y_t^{n+1} - Y_t^n|^2$ ,  $\beta \geq 0$ , we have

$$\begin{aligned} e^{\beta t}|Y_t^{n+1} - Y_t^n|^2 &= e^{\beta T}|h(X_T^{n+1}) - h(X_T^n)|^2 - 2 \int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n)dW_s \\ &\quad - \int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds - \int_t^T \beta e^{\beta s}(Y_s^{n+1} - Y_s^n)^2 ds \\ &\quad + 2 \int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n) [g_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}) - g_s(X_s^n, Y_s^n, Z_s^n)] ds. \end{aligned}$$

Hence, due to the condition (A3) and the boundedness of  $(Z^n)$ , it holds

$$\begin{aligned} &e^{\beta t}|Y_t^{n+1} - Y_t^n|^2 + \int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds \\ &\leq e^{\beta T}|h(X_T^{n+1}) - h(X_T^n)|^2 - 2 \int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n)dW_s \\ &\quad - \int_t^T \beta e^{\beta s}(Y_s^{n+1} - Y_s^n)^2 ds + 2 \int_t^T e^{\beta s} \rho(M) |Y_s^{n+1} - Y_s^n| |Z_s^{n+1} - Z_s^n| ds \\ &\quad + 2 \int_t^T e^{\beta s} k_7 |Y_s^{n+1} - Y_s^n| |X_s^{n+1} - X_s^n| ds + 2 \int_t^T e^{\beta s} k_4 |Y_s^{n+1} - Y_s^n|^2 ds. \end{aligned}$$

With some positive constants  $\alpha_1, \alpha_2$ , it follows from (A3) and Young's inequality that

$$\begin{aligned} &e^{\beta t}|Y_t^{n+1} - Y_t^n|^2 + \int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds \leq e^{\beta T} k_5^2 |X_T^{n+1} - X_T^n|^2 \\ &\quad - 2 \int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n)dW_s + \alpha_2 \int_t^T e^{\beta s} |X_s^{n+1} - X_s^n|^2 ds \\ &\quad + \left( \frac{(\rho(M))^2}{\alpha_1} + \frac{k_3^2}{\alpha_2} + 2k_4 - \beta \right) \int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)^2 ds + \alpha_1 \int_t^T e^{\beta s} |Z_s^{n+1} - Z_s^n|^2 ds. \end{aligned} \tag{4.3}$$

Letting  $\beta = \frac{(\rho(M))^2}{\alpha_1} + \frac{k_7^2}{\alpha_2} + 2k_8$  and taking expectation on both sides above, we have

$$\begin{aligned} E \left[ e^{\beta t} |Y_t^{n+1} - Y_t^n|^2 \right] + E \left[ \int_t^T e^{\beta s} (Z_s^{n+1} - Z_s^n)^2 ds \right] &\leq e^{\beta T} k_5^2 E [|X_T^{n+1} - X_T^n|^2] \\ &+ \alpha_1 E \left[ \int_t^T e^{\beta s} |Z_s^{n+1} - Z_s^n|^2 ds \right] + \alpha_2 E \left[ \int_t^T e^{\beta s} |X_s^{n+1} - X_s^n|^2 ds \right]. \end{aligned}$$

Putting  $\alpha_1 = \frac{1}{2}$  and  $\alpha_2 = 1$ , the previous estimate yields

$$E \left[ \int_0^T e^{\beta s} (Z_s^{n+1} - Z_s^n)^2 ds \right] \leq 2e^{\beta T} k_5^2 E [|X_T^{n+1} - X_T^n|^2] + 2E \left[ \int_0^T e^{\beta s} |X_s^{n+1} - X_s^n|^2 ds \right].$$

Next, taking conditional expectation with respect to  $\mathcal{F}_t$  in (4.3) gives

$$\begin{aligned} e^{\beta t} |Y_t^{n+1} - Y_t^n|^2 + E \left[ \int_t^T e^{\beta s} (Z_s^{n+1} - Z_s^n)^2 ds \middle| \mathcal{F}_t \right] &\leq e^{\beta T} k_5^2 E [|X_T^{n+1} - X_T^n|^2 | \mathcal{F}_t] \\ &+ \alpha_1 E \left[ \int_t^T e^{\beta s} |Z_s^{n+1} - Z_s^n|^2 ds \middle| \mathcal{F}_t \right] + \alpha_2 E \left[ \int_t^T e^{\beta s} |X_s^{n+1} - X_s^n|^2 ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

Thus, by Burkholder-Davis-Gundy's inequality, with a positive constant  $c_1$  and  $\alpha_1 = \frac{1}{2}$ ,  $\alpha_2 = 1$ , we have

$$\begin{aligned} E \left[ \sup_{0 \leq t \leq T} e^{\beta t} |Y_t^{n+1} - Y_t^n|^2 \right] &\leq c_1 e^{\beta T} k_5^2 E [|X_T^{n+1} - X_T^n|^2] \\ &+ c_1 \frac{1}{2} E \left[ \int_0^T e^{\beta s} |Z_s^{n+1} - Z_s^n|^2 ds \right] + c_1 E \left[ \int_0^T e^{\beta s} |X_s^{n+1} - X_s^n|^2 ds \right] \\ &\leq 2c_1 e^{\beta T} k_5^2 E [|X_T^{n+1} - X_T^n|^2] + 2c_1 E \left[ \int_0^T e^{\beta s} |X_s^{n+1} - X_s^n|^2 ds \right]. \end{aligned}$$

It now follows from (4.2) that

$$\begin{aligned} E \left[ \sup_{0 \leq t \leq T} |Y_t^{n+1} - Y_t^n|^2 \right] + E \left[ \int_0^T (Z_s^{n+1} - Z_s^n)^2 ds \right] \\ \leq 8(c_1 + 1)e^{\beta T} (k_5^2 + T) T^2 k_2^2 E \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^{n-1}|^2 \right]. \end{aligned}$$

Taking  $T$  small enough so that

$$8(c_1 + 1)e^{\beta T}(k_5^2 + T)T^2k_2^2 \leq \frac{1}{2},$$

we obtain that  $(X^n, Y^n, Z^n)$  is a Cauchy sequence in  $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$ . Thus, it suffices to define  $2_{k,\lambda,d,l}^2$  by the conditions

$$\begin{cases} 2T^2k_1^2 \leq \frac{1}{2} \\ 8(c_1 + 1)e^{\beta T}(k_5^2 + T)T^2k_2^2 \leq \frac{1}{2}. \end{cases}$$

By continuity of  $b, g$  and  $h$  we have the existence of a solution  $(X, Y, Z)$  in  $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$  of FBSDE (1.1) and it follows from the boundedness of  $(Z^n)$  that  $|Z_t| \leq M$ . The uniqueness in  $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  follows from the boundedness of  $Z$  and by repeating the above arguments on the difference of two solutions.

*Step 3:* If one of the functions  $b, g$  or  $h$  is not differentiable, we apply the technique of the proof of Theorem 3.1. Namely, we use an approximation by the smooth functions defined as follows: For  $n \in \mathbb{N}$ , let  $\beta_n^1, \beta_n^2$  and  $\beta_n^3$  be nonnegative  $C^\infty$  functions with support on  $\{x \in \mathbb{R}^m : |x| \leq \frac{1}{n}\}$ ,  $\{x \in \mathbb{R}^{m+l} : |x| \leq \frac{1}{n}\}$  and  $\{x \in \mathbb{R}^{m+l+l \times d} : |x| \leq \frac{1}{n}\}$  respectively, and satisfying  $\int_{\mathbb{R}^m} \beta_n^1(r)dr = 1$ ,  $\int_{\mathbb{R}^{m+l}} \beta_n^2(r)dr = 1$  and  $\int_{\mathbb{R}^{m+l+l \times d}} \beta_n^3(r)dr = 1$ . We define the convolutions

$$\begin{aligned} b_t^n(x, y) &:= \int_{\mathbb{R}^{m+l}} b_t(x', y') \beta_n^2(x' - x, y' - y) dx' dy', \quad h^n(x) := \int_{\mathbb{R}^m} h(x') \beta_n^1(x' - x) dx', \\ g^n(u, x, y, z) &:= \int_{\mathbb{R}^{m+l+l \times d}} g(u, x', y', z') \beta_n^3(x' - x, y' - y, z' - z) dx' dy' dz'. \end{aligned}$$

It is easy to check that  $b^n$  satisfies (A1) with the constants  $k_1, k_2$  and  $2\lambda_1$  and that  $g^n$  and  $h^n$  satisfy (A4) - (A5) and (A3), respectively, with the same constants. From Steps 1 and 2, there exists a positive constant  $\bar{C}_{k,\lambda,l,d}$  independent of  $n$  such that if  $T \leq \bar{C}_{k,\lambda,l,d}$ , FBSDE (1.1) with parameters  $(b^n, h^n, g^n)$  admits a unique solution  $(X^n, Y^n, Z^n) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  and

$$|Z_t^n| \leq M.$$

By the Lipschitz continuity conditions on  $b$  and  $h$  and the locally Lipschitz condition of  $g$ , the sequences  $(b^n)$  and  $(h^n)$  converge uniformly to  $b$  and  $h$  on  $\mathbb{R}^{m+l}$  and  $\mathbb{R}^m$ , respectively, and  $(g^n)$  converges to  $g$  uniformly on  $\mathbb{R}^{m+l} \times \Lambda$  for any compact subset  $\Lambda$  of  $\mathbb{R}^{l \times d}$ . Combining these uniform convergences with the boundedness of  $Z^n$ , similar to above, we can show that there exists a constant  $\tilde{C}_{k,\lambda,l,d}$  depending only on  $k_1, k_2, k_3, k_4, k_5, \lambda_2, l, d$  such that if  $T \leq \tilde{C}_{k,\lambda,l,d}$ ,  $(X^n, Y^n, Z^n)$  is a Cauchy sequence in the Banach space  $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l \times d})$ .

In fact, for any  $m, n \in \mathbb{N}$ , using Cauchy-Schwarz' inequality we have

$$|X_t^n - X_t^m|^2 \leq T \int_0^T |b_u^n(X_u^n, Y_u^n) - b_u^m(X_u^m, Y_u^m)|^2 du.$$

Thus, taking the supremum with respect to  $t$  and then expectation on both sides give

$$\begin{aligned}
& \|X^n - X^m\|_{\mathcal{S}^2(\mathbb{R}^m)}^2 \\
& \leq 3T \int_0^T (|b_u^n(X_u^n, Y_u^n) - b_u(X_u^n, Y_u^n)|^2 + |b_u^m(X_u^m, Y_u^m) - b_u(X_u^m, Y_u^m)|^2 \\
& \quad + |b_u(X_u^n, Y_u^n) - b_u(X_u^m, Y_u^m)|^2) du \\
& \leq 3T \int_0^T (|b_u^n(X_u^n, Y_u^n) - b_u(X_u^n, Y_u^n)|^2 + |b_u^m(X_u^m, Y_u^m) - b_u(X_u^m, Y_u^m)|^2) du \\
& \quad + 3k_1^2 T^2 \|X^n - X^m\|_{\mathcal{S}^2(\mathbb{R}^m)}^2 + 3k_2^2 T^2 \|Y^n - Y^m\|_{\mathcal{S}^2(\mathbb{R}^l)}^2
\end{aligned} \tag{4.4}$$

where the second inequality follows from (A1). On the other hand, applying Itô's formula as in Step 2, one has

$$\begin{aligned}
& |Y_t^m - Y_t^n|^2 + \int_t^T |Z_u^n - Z_u^m|^2 du \\
& \leq |h^n(X_T^n) - h(X_T^n)|^2 + |h^m(X_T^m) - h(X_T^m)|^2 + k_5^2 |X_T^n - X_T^m|^2 \\
& \quad - 2 \int_t^T (Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n) dW_s \\
& \quad + \int_t^T |Y_u^n - Y_u^m| (|g_u^n(X_u^n, Y_u^n, Z_u^n) - g_u(X_u^n, Y_u^n, Z_u^n)| \\
& \quad + |g_u^m(X_u^m, Y_u^m, Z_u^m) - g_u(X_u^m, Y_u^m, Z_u^m)| + k_3 |X_u^n - X_u^m| + k_4 |Y_u^n - Y_u^m| \\
& \quad + \rho(M) |Z_u^n - Z_u^m|).
\end{aligned} \tag{4.5}$$

Taking expectation, due to Young's inequality we have

$$\begin{aligned}
& \|Z^n - Z^m\|_{\mathcal{H}^2(\mathbb{R}^l \times d)}^2 \\
& \leq E[|h^n(X_T^n) - h(X_T^n)|^2] + E[|h^m(X_T^m) - h(X_T^m)|^2] + (k_5^2 + \frac{1}{2} T k_3^2) \|X^n - X^m\|_{\mathcal{S}^2(\mathbb{R}^m)}^2 \\
& \quad + \frac{1}{2} T \|Y^n - Y^m\|_{\mathcal{S}^2(\mathbb{R}^l)}^2 + \frac{1}{2} \int_0^T |g_u^n(X_u^n, Y_u^n, Z_u^n) - g_u(X_u^n, Y_u^n, Z_u^n)|^2 \\
& \quad + |g_u^m(X_u^m, Y_u^m, Z_u^m) - g_u(X_u^m, Y_u^m, Z_u^m)|^2 du \\
& \quad + \frac{1}{2} T k_4^2 \rho^2(M) \|Y^n - Y^m\|_{\mathcal{S}^2(\mathbb{R}^l)}^2 + \frac{1}{2} \|Z^n - Z^m\|_{\mathcal{H}^2(\mathbb{R}^l \times d)}^2.
\end{aligned}$$

On the other hand, taking conditional expectation in (4.5) and then the supremum with respect



to  $t$  and then expectation on both sides, we have due to Young's inequality

$$\begin{aligned}
& \|Y^n - Y^m\|_{\mathcal{S}^2(\mathbb{R}^l)}^2 \\
& \leq E[|h^n(X_T^n) - h(X_T^n)|^2] + E[|h^m(X_T^m) - h(X_T^m)|^2] + k_5^2 \|X^n - X^m\|_{\mathcal{S}^2(\mathbb{R}^m)}^2 \\
& \quad + \frac{1}{2} T \|Y^n - Y^m\|_{\mathcal{S}^2(\mathbb{R}^l)}^2 + \frac{1}{2} \int_0^T |g_u^n(X_u^n, Y_u^n, Z_u^n) - g_u(X_u^n, Y_u^n, Z_u^n)|^2 \\
& \quad + |g_u^m(X_u^m, Y_u^m, Z_u^m) - g_u(X_u^m, Y_u^m, Z_u^m)|^2 du \\
& \quad + \frac{1}{2} T k_3^2 \|X^n - X^m\|_{\mathcal{S}^2(\mathbb{R}^m)}^2 + \frac{1}{2} T k_4^2 \rho^2(M) \|Y^n - Y^m\|_{\mathcal{S}^2(\mathbb{R}^l)}^2 + \frac{1}{2} \|Z^n - Z^m\|_{\mathcal{H}^2(\mathbb{R}^l \times d)}^2.
\end{aligned}$$

Combining (4.4) and (4.1) we observe that if  $T$  is small enough so that

$$\begin{cases} 3k_1^2 T^2 \leq \frac{1}{2} \\ \frac{1}{2} T + 3k_5^2 k_2^2 T^2 + \frac{3}{2} T^3 k_3^2 k_2^2 + \frac{1}{2} T k_4^2 \rho^2(M) \leq \frac{1}{2} \end{cases}$$

then, the uniform convergence of  $(b^n)$ ,  $g^n$  and  $(h^n)$  to  $b$ ,  $g$  and  $h$  ensure that  $(X^n, Y^n, Z^n)$  is a Cauchy sequence. The verification that the limit  $(X, Y, Z)$  of the sequence  $(X^n, Y^n, Z^n)$  solves the FBSDE (1.1) uses continuity of the functions  $b$ ,  $h$  and  $g$ , and that  $|Z_t| \leq M$  is a consequence of the boundedness of  $(Z^n)$ . Taking  $C_{k,\lambda,l,d} := \tilde{C}_{k,\lambda,l,d} \wedge \bar{C}_{k,\lambda,l,d}$  concludes the proof.  $\square$

Due to Theorem 2.1 above, our global existence result now follows from a pasting procedure.

## 4.2 Proof of Theorem 2.2

If  $T \leq C_{k,\lambda,l,d}$ , then the result follows from Theorem 2.1.

Assume  $T > C_{k,\lambda,l,d}$  and let  $\tilde{h}_M : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function whose derivative is bounded by 1 and such that  $\tilde{h}'_M(a) = 1$  for all  $-M \leq a \leq M$  and

$$\tilde{h}_M(a) = \begin{cases} (M+1) & \text{if } a > M+2 \\ a & \text{if } |a| \leq M \\ -(M+1) & \text{if } a < -(M+2). \end{cases}$$

An example of such a function is given by

$$\tilde{h}_M(a) = \begin{cases} (-M^2 + 2Ma - a(a-4)) / 4 & \text{if } a \in [M, M+2] \\ (M^2 + 2Ma + a(a+4)) / 4 & \text{if } a \in [-(M+2), -M], \end{cases}$$

see Imkeller and Reis [18]. By the assumptions (A3) the function  $\tilde{g} : [0, T] \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^{l \times d} \rightarrow \mathbb{R}$  defined by

$$\tilde{g}_t(x, y, z) := g_t(x, y, h_M(z)) \tag{4.6}$$

with  $h_M(z) := (\tilde{h}_M(z^{ij}))_{ij}$  is Lipschitz continuous in all variables. Thus, it follows from Delarue [7, Theorem 2.6] that the equation

$$\begin{cases} \tilde{X}_t = x + \int_0^t b_u(\tilde{X}_u, \tilde{Y}_u) du + \int_0^t \sigma_u dW_u \\ \tilde{Y}_t = h(\tilde{X}_T) + \int_t^T \tilde{g}_u(\tilde{X}_u, \tilde{Y}_u, \tilde{Z}_u) du - \int_t^T \tilde{Z}_u dW_u, \quad t \in [0, T] \end{cases} \tag{4.7}$$

admits a unique solution  $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$ . Moreover, there exists a Lipschitz continuous function  $\theta : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^l$  bounded by a constant  $K$  such that  $\tilde{Y}_t = \theta(t, \tilde{X}_t)$  for all  $t \in [0, T]$ . In fact, for every  $x, x' \in \mathbb{R}^d$ ,  $t \in [0, T]$  and  $i = 1, \dots, l$  we have

$$\begin{aligned} & \theta(t, \tilde{X}_t^x) - \theta(t, \tilde{X}_t^{x'}) \\ &= h^i(\tilde{X}_T^x) - h^i(\tilde{X}_T^{x'}) + \int_t^T g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'}) du - \int_t^T \tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i} dW_u \\ &= h^i(\tilde{X}_T^x) - h^i(\tilde{X}_T^{x'}) + \int_t^T \frac{g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'})}{\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}} (\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}) 1_{\{|\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}| \neq 0\}} du \\ &\quad + \int_t^T (g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'})) 1_{\{|\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}| = 0\}} du - \int_t^T \tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i} dW_u. \end{aligned}$$

Thus, Girsanov's theorem yields

$$\begin{aligned} & \left| \theta^i(t, \tilde{X}_t^x) - \theta^i(t, \tilde{X}_t^{x'}) \right| \leq \\ & E^{Q^i} \left[ \left| h^i(\tilde{X}_T^x) - h^i(\tilde{X}_T^{x'}) \right| + \int_t^T \left| g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'}) \right| 1_{\{|\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}| = 0\}} du \mid \mathcal{F}_t \right] \\ & \leq E^{Q^i} \left[ k_5 |\tilde{X}_T^x - \tilde{X}_T^{x'}| + \int_t^T \left( k_3 |\tilde{X}_u^x - \tilde{X}_u^{x'}| + k_4 |\tilde{Y}_u^x - \tilde{Y}_u^{x'}| \right) du \mid \mathcal{F}_t \right] \end{aligned}$$

where  $Q^i$  is the probability measure given by

$$\frac{dQ^i}{dP} = \mathcal{E} \left( \frac{g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'})}{\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}} 1_{\{|\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}| \neq 0\}} \cdot W \right)_T.$$

By (A4') and boundedness of  $\tilde{Y}^x$  and  $\tilde{Y}^{x'}$  is well defined. Since by Gronwall's lemma we have

$$|\tilde{X}_s^x - \tilde{X}_s^{x'}| \leq (|\tilde{X}_t^x - \tilde{X}_t^{x'}| + k_2 \int_s^T |\tilde{Y}_u^x - \tilde{Y}_u^{x'}| du) e^{k_1 T}, \quad s \in [t, T],$$

it holds

$$\begin{aligned} & \left| \theta^i(t, \tilde{X}_t^x) - \theta^i(t, \tilde{X}_t^{x'}) \right| \\ & \leq E^{Q^i} \left[ e^{k_1 T} (k_5 + T k_3) |\tilde{X}_t^x - \tilde{X}_t^{x'}| + (k_2 k_3 T e^{k_1 T} + k_4 + k_2 k_5 e^{k_1 T}) \int_t^T |\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})| du \mid \mathcal{F}_t \right]. \end{aligned}$$

Hence,  $\left| \theta^i(t, \tilde{X}_t^x) - \theta^i(t, \tilde{X}_t^{x'}) \right| \leq u_t$  where  $u_t$  is the solution of the ODE

$$u_t = e^{k_1 T} (k_5 + Tk_3) l |\tilde{X}_t^x - \tilde{X}_t^{x'}| + (k_2 k_3 T e^{k_1 T} + k_4 + k_2 k_5 e^{k_1 T}) \int_t^T l u_s du$$

which is given by

$$u_t = e^{k_1 T} (k_5 + Tk_3) l |\tilde{X}_t^x - \tilde{X}_t^{x'}| \exp \left( (k_2 k_3 T e^{k_1 T} + k_4 + k_2 k_5 e^{k_1 T}) l (T - t) \right).$$

Thus,

$$\left| \theta(t, \tilde{X}_t^x) - \theta(t, \tilde{X}_t^{x'}) \right| \leq K_5 |\tilde{X}_t^x - \tilde{X}_t^{x'}|,$$

with  $K_5 := \sqrt{l} e^{k_1 T} (k_5 + Tk_3) l \exp \left( (k_2 k_3 T e^{k_1 T} + k_4 + k_2 k_5 e^{k_1 T}) l T \right)$  which show that  $\theta$  is a Lipschitz function and the Lipschitz coefficient does not depend on the bound  $M$  of  $Z$ .

Let  $\bar{C}_{k,\lambda,l,d}$  be the constant  $C_{k,\lambda,l,d}$  with  $k_5$  replaced by  $K_5$  and put  $N = \lfloor T/\bar{C}_{k,\lambda,l,d} \rfloor$ , where  $\lfloor a \rfloor$  denotes the integer part of  $a$ , and  $t_i := i\bar{C}_{k,\lambda,l,d}$ ,  $i = 0, \dots, N$  and  $t_{N+1} = T$ . Since  $t_1 \leq \bar{C}_{k,\lambda,l,d}$ , by Theorem 2.1 the FBSDE

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u) du + \int_0^t \sigma_u dW_u \\ Y_t = \theta(t_1, X_{t_1}) + \int_t^{t_1} g_u(X_u, Y_u, Z_u) du - \int_t^{t_1} Z_u dW_u, \quad t \in [0, t_1] \end{cases}$$

admits a unique solution  $(X^1, Y^1, Z^1)$  such that  $|Z_t^1| \leq \bar{M}$  with  $\bar{M} = 4\lambda_2 K_5 \sqrt{dl}$  for all  $t \in [0, t_1]$ . Therefore,  $(X^1, Y^1, Z^1)1_{[0, t_1]} = (\tilde{X}, \tilde{Y}, \tilde{Z})1_{[0, t_1]}$ . Similarly, we obtain a family  $(X^i, Y^i, Z^i)$  of solutions of the FBSDEs

$$\begin{cases} X_t = \tilde{X}_{t_{i-1}} + \int_{t_{i-1}}^t b_u(X_u, Y_u) du + \int_{t_{i-1}}^t \sigma_u dW_u \\ Y_t = \theta(t_i, X_{t_i}) + \int_t^{t_i} g_u(X_u, Y_u, Z_u) du - \int_t^{t_i} Z_u dW_u, \quad t \in [t_{i-1}, t_i] \end{cases}$$

such that  $(X^i, Y^i, Z^i)1_{[t_{i-1}, t_i]} = (\tilde{X}, \tilde{Y}, \tilde{Z})1_{[t_{i-1}, t_i]}$ ,  $i = 1, \dots, N+1$ . Define

$$X := \sum_{i=1}^{N+1} X^i 1_{[t_{i-1}, t_i]}; \quad Y := \sum_{i=1}^{N+1} Y^i 1_{[t_{i-1}, t_i]} \quad \text{and} \quad Z := \sum_{i=1}^{N+1} Z^i 1_{[t_{i-1}, t_i]}.$$

Then,  $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  is the unique solution of the FBSDE (1.1) satisfying  $|Z_t| \leq \bar{M}$  for all  $t \in [0, T]$ . In fact, it is clear that  $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  as a finite sum of elements of the same space. Let  $t \in [0, T]$  and  $i = 1, \dots, N+1$  such that  $t \in [t_{i-1}, t_i]$ . We have

$$\begin{aligned} x + \int_0^t b_u(X_u) du + \int_0^t \sigma_u du &= x + \sum_{j=1}^i \left( \int_{t_{j-1}}^{t_j \wedge t} b_u(X_u^j) du + \int_{t_{j-1}}^{t_j \wedge t} \sigma_u dW_u \right) \\ &= X_t^i = X_t \end{aligned}$$

and

$$\begin{aligned} h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u) du - \int_t^T Z_u dW_u \\ = h(X_T^{N+1}) + \sum_{j=i}^{N+1} \left( \int_{t_{j-1} \vee t}^{t_j} g_u(X_u^j, Y_u^j, Z_u^j) du - \int_{t_{j-1} \vee t}^{t_j} Z_u^j dW_u \right) = Y_t^i = Y_t. \end{aligned}$$

That is,  $(X, Y, Z)$  satisfies Equation (1.1).  $\square$

### 4.3 Proof of Proposition 2.4

By Theorem 2.2 the FBSDE

$$\begin{cases} \tilde{X}_t &= x + \int_0^t \tilde{b}_s(\tilde{X}_s, \tilde{Y}_s) ds + \int_0^t \sigma_s dW_s \\ \tilde{Y}_t &= \tilde{h}(\tilde{X}_T) + \int_t^T \tilde{g}_s(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s, \quad t \in [0, T] \end{cases}$$

has a unique global solution  $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  such that  $|\tilde{Z}_t| \leq \tilde{M}$  for some constant  $\tilde{M} \geq 0$ . Let  $X_t = \tilde{X}_t$ ,  $Y_t = \Gamma^{-1} \tilde{Y}_t$ ,  $Z_t = \Gamma^{-1} \tilde{Z}_t$ , we obtain that  $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l \times d})$  such that  $|Z_t| \leq \bar{M}$  for some constant  $\bar{M} \geq 0$ . Moreover,  $(X, Y, Z)$  satisfies the FBSDE (1.1). This completes the proof.  $\square$

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